Observer-Based Controller for Disturbance Decoupling of Max-plus Linear Systems with Applications to a High Throughput Screening System in Drug Discovery

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Abstract— **Max-plus linear systems are often used to model timed discrete-event systems, which represent system operations as discrete sequences of events in time. This paper presents the observer-based controller to solve the disturbance decoupling problem for max-plus linear systems where only estimations of system states are available for the controller. This observer-based controller leads to a greater control input than the one obtained with the output feedback strategy based on just-in-time criterion. A high throughput screening system in drug discovery illustrates this main result by showing that the scheduling obtained from the observer-based controller solving the disturbance decoupling problem is better than the scheduling obtained from the output feedback controller.**

I. Introduction

Max-plus linear systems ([1], [3], [20]) are used to model for timed discrete-event systems, which represent the system operations as discrete sequences of events in time. The main advantage of max-plus linear systems is incorporating the traditional linear system theory for the nonlinear concurrent behaviors in discrete-event systems. Over the past three decades, many fundamental problems for max-plus linear systems have been studied by researchers, for example, controllability ([21]), observability ([11]), feedback control ([22]) and model reference control ([18]). However, the geometric theory for max-plus linear systems introduced in ([5]) has not been well established as the traditional linear systems ([2], [25]). Only a few existing research results on fundamental concepts and problems in the geometric control of conventional dynamic systems are generalized to max-plus linear systems, such as computation of different controlled invariant sets ([8], [13], [19]) and the disturbance decoupling problem ([15]).

This paper reports upon further investigations on the disturbance decoupling problem (DDP) and modified disturbance decoupling problem (MDDP) ([9], [23]) for max-plus linear systems. For a manufacturing system, solving the DDP means that the outputs will be delayed more than the delays caused by the disturbances. From a practical point of view, it would be interesting to ask the question as whether there exists a controller such that the system is not disturbed more than the delays caused by the disturbances. MDDP, on the other hand, is to find a control such that the output signals generated by the control will not be greater than the output signals caused by the disturbances. In the previous work, state-feedback controllers and output feedback controllers [24] have been developed to solve the DDP and MDDP, respectively. This paper presents an observer-based controller for max-plus linear systems using the observer introduced in ([10], [11], [12]), an analogy with the classical Luenberger observer [17] for linear systems. The main result is that this observer-based controller leads to a greater control input than the one obtained with the output feedback strategy solving the MDDP and DDP, in spite of the lack of sensors. For instance, in a manufacturing setting, the observer-based controller would provide a better scheduling by starting the process later than the output feedback controller, while ensuring the same finishing time of the output parts. This scheduling would allow users to load the raw parts later rather than earlier to avoid unnecessary congestions in manufacturing lines.

This paper is organized as the following. Section II presents some algebraic tools concerning max-plus algebraic structures. Section III presents the definitions of DDP and MDDP for max-plus linear systems. Section IV reviews the state-feedback controller and the output feedback controller solving DDP and MDDP, respectively, in [24]. Section V reviews the max-plus observer in [11]. Section VI presents the observer-based controller and compares its differences in performance with the output feedback controller and the state-feedback controller in [24]. Section VII illustrates the main results using a high throughput screening example in drug discovery. An observer-based controller is constructed and proved to have a better performance than the output feedback controller. Section VIII concludes this paper with future research directions.

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II. Mathematical Preliminaries

Definition 1: A *semiring* is a set $\mathcal{S}$, equipped with two operations $\oplus$ and $\otimes$, such that $(\mathcal{S}, \oplus)$ is a commutative monoid (the zero element will be denoted ε), $(\mathcal{S}, \otimes)$ is a monoid (the unit element will be denoted e), operation $\otimes$ is right and left distributive over $\oplus$, and ε is absorbing for the product (*i.e.* $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon, \forall a$).

A semiring $\mathcal{S}$ is *idempotent* if $a \oplus a = a$ for all $a \in \mathcal{S}$. In an idempotent semiring $\mathcal{S}$, operation $\oplus$ induces a partial order relation $a \succeq b \iff a = a \oplus b, \quad \forall a, b \in \mathcal{S}$. Then, $a \vee b = a \oplus b$. An idempotent semiring $\mathcal{S}$ is *complete* if the sum of infinite numbers of terms is always defined, and if multiplication distributes over infinite sums too. In particular, the sum of all the elements of the idempotent semiring is denoted $\top$ (for "top"). In this paper, we denote $\overline{\mathbb{Z}}_{\max} = (\mathbb{Z} \cup \{-\infty, +\infty\}, max, -\infty, +, 0)$, where $\varepsilon = -\infty$ is the neutral element to max and $e = 0$ is the neutral element to $+$. the integer max-plus semiring. A non empty subset $\mathcal{B}$ of a semiring $\mathcal{S}$ is a subsemiring of $\mathcal{S}$ if for all $a, b \in \mathcal{B}$ we have $a \oplus b \in \mathcal{B}$ and $a \otimes b \in \mathcal{B}$.

Definition 2: A mapping $f : \mathcal{S} \to \mathcal{S}$, where $\mathcal{S}$ is a complete idempotent semiring, is residuated if and only if $f(\varepsilon) = \varepsilon$ and f is lower-semicontinuous, that is,

$$f\left(\bigoplus_{i \in I} a_i\right) = \bigoplus_{i \in I} f(a_i),$$

for any (finite or infinite) set I. The mapping f is said to be residuated and $f^\sharp$ is called its residual.

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When f is residuated, $f^\sharp$ is the unique order preserving mapping such that

$$f \circ f^\sharp \preceq \mathsf{Id} \qquad f^\sharp \circ f \succeq \mathsf{Id}, \tag{1}$$

where Id is the identity mapping from $\mathcal{S}$ to $\mathcal{S}$. It is straightforward that : $L_a : \mathcal{S} \to \mathcal{S}, x \mapsto ax$ and $R_a : \mathcal{S} \to \mathcal{S}, x \mapsto xa$ are lower semi-continuous. Therefore these mappings are both residuated *i.e.*, $L_a(x) \preceq b$ (resp. $R_a(x) \preceq b$) admits a greatest solution, then the following notations are considered :

$$L_a^\sharp(b) = a \backslash\!\!\circ b = \bigoplus \{x | ax \preceq b\} \text{ and}$$
$$R_a^\sharp(b) = b /\!\!\circ a = \bigoplus \{x | xa \preceq b\}, \quad \forall a, b \in \mathcal{S}.$$

All these results admit a natural extension to the matrix case, where the sum and product of matrices are defined with the same rules as in classical theory (see [1]). Over complete idempotent semiring, the implicit equation $x = ax \oplus b$ admits $x = a^* b$ as the least solution, where $a^* = \oplus_{i \in \mathbb{N}} a_i$ (Kleene star operator) with $a_0 = e$.

Property 1: ([20]) Given a complete semiring $\mathcal{S}$, and two matrices $A \in \mathcal{S}^{p \times n}$, $B \in \mathcal{S}^{n \times p}$, the following equations hold :

$$A \backslash\!\!\circ A \;=\; (A \backslash\!\!\circ A)^*, \; B /\!\!\circ B = (B /\!\!\circ B)^*. \tag{2}$$

III. Disturbance Decoupling in Max-Plus Linear Systems

A max-plus linear system can be described by the following

$$\begin{aligned} x(k) &= Ax(k-1) \oplus Bu(k) \oplus Sq(k), \\ y(k) &= Cx(k), \end{aligned} \tag{3}$$

where the state is $x(k) \in \overline{\mathbb{Z}}^n_{\max}$, the input is $u(k) \in \overline{\mathbb{Z}}^p_{\max}$, the disturbance is $q(k) \in \overline{\mathbb{Z}}^r_{\max}$, and the output is $y(k) \in \overline{\mathbb{Z}}^q_{\max}$, for $k \in \mathbb{Z}$. This kind of system makes it possible to describe the behaviors of timed-event graphs(TEGs)[1] by associating to each transition x_i a firing date sequence $x_i(k) \in \overline{\mathbb{Z}}_{\max}$, and predict the system evolution thanks to Eq. (3).

Definition 3: System (3) is called **disturbance decoupled** by an external control if and only if any disturbance signal will not affect the system output $y(k)$ for all $k \in \mathbb{Z}$, that is, the output signals $y(k)$ remain the same as the output signals of the undisturbed system, for all $k \in \mathbb{Z}$.

In manufacturing systems, for example, when the system breaks down for one hour, the control will delay the system operation more than one hour in order to achieve the DDP. In practical scenarios, production lines need to resume as soon as the system breakdown is fixed. Hence, a modified DDP is introduced in order to find the optimal just-in-time controls such that the system will start running as soon as possible once the system breakdown is recovered. The MDDP means that, for manufacturing systems, the controls will delay the starting dates of the process such that the finishing date of the output parts would be sooner than the finishing dates after the system breakdown.

Definition 4: The max-plus linear system described in Eq. (3) is called **modified disturbance decoupled** by an external control if and only if the system output signals generated by the controls will not be greater than the output signals generated by only the disturbances.

Control signals that can solve DDP ad MDDP, respectively, are loop controls $u(k) = v(k)$, state-feedback controls $u_K(k) = Kx(k-1) \oplus v(k)$, or output feedback controls $u_F(k) = Fy(k) \oplus v(k)$. After introducing the max-plus observer in the next section, the observer-based control $u_M(k) = M\hat{x}(k-1) \oplus v(k)$, where $\hat{x}$ is an estimation of the original state x, will be presented. Furthermore, performance of different types of controls solving DDP and MDDP will be compared and illustrated in the remainder of the paper.

[1]Timed-event graphs (TEGs) are timed Petri nets where each place has exactly one upstream transition and one downstream transition.

IV. Solving DDP and MDDP using Event-Domain Approach

A. Event-Domain Representation

For a state equation in Eq. (3), each increasing sequence $\{x(k)\}$, it is possible to define the transformation $X(\gamma) = \bigoplus_{k \in \mathbb{Z}} x(k)\gamma^k$ where γ is a backward shift operator in event domain (*i.e.*, $Y(\gamma) = \gamma X(\gamma) \iff \{y(k)\} = \{x(k-1)\}$, (see [1], p. 228). This transformation is analogous to the z-transform used in discrete-time classical control theory and the formal series $X(\gamma)$ is a synthetic representation of the trajectory $x(k)$. The set of the formal power series in γ is denoted by $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ and constitutes an idempotent semiring. Therefore, the state equation in Eq. (3) becomes a polynomial equation or an event-domain representation,

$$\begin{aligned} X(\gamma) &= \overline{A}X(\gamma) \oplus BU(\gamma) \oplus SQ(\gamma), \text{ where } \overline{A} = \gamma A, \\ Y(\gamma) &= CX(\gamma), \end{aligned} \tag{4}$$

where the state $X(\gamma) \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^n$, the output $Y(\gamma) \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^q$, the input $U(\gamma) \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^p$, and the disturbance $Q(\gamma) \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^r$, and matrices $\overline{A} \triangleq \gamma A \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{n \times n}$, $B \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{n \times p}$, $C \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{q \times n}$ and $S \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{n \times r}$ represent the link between transitions. According to the state equation (4), the evolution of the system is

$$\begin{aligned} X(\gamma) &= \overline{A}^* BU(\gamma) \oplus \overline{A}^* SQ(\gamma) \\ Y(\gamma) &= C\overline{A}^* BU(\gamma) \oplus C\overline{A}^* SQ(\gamma). \end{aligned} \tag{5}$$

The trajectories $U(\gamma)$ and $Y(\gamma)$ can be related ([1], p. 243) by the equation $Y(\gamma) = H(\gamma)U(\gamma)$, where $H(\gamma) = C\overline{A}^* B \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{q \times p}$ is called the transfer matrix of the TEG. Entries of matrix H are periodic series ([1], p. 260) in the idempotent semiring, usually represented by $p(\gamma) \oplus q(\gamma)(\tau\gamma^\nu)^*$, where $p(\gamma)$ is a polynomial representing the transient behavior, $q(\gamma)$ is a polynomial corresponding to a pattern which is repeated periodically, the period being given by the monomial $(\tau\gamma^\nu)$. The disturbances are uncontrollable inputs acting on the system internal's state, which model events that block the system, $e.g.$ machine breakdown, uncontrollable component supply through matrix S, and $C\overline{A}^* S \in \left(\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]\right)^{q \times r}$ is the transfer function between the disturbances and outputs.

B. Solving DDP and MDDP by an Open-Loop Control

The objective of the MDDP is to find the greatest open-loop control $U(\gamma)$ such that the output trajectories will not be disturbed more than the disturbance signals have acted on the system. Formally, according to Definition 4, this means to find the greatest control, $U(\gamma)$, such that the following equation holds,

$$\begin{aligned} C\overline{A}^* BU(\gamma) \oplus C\overline{A}^* SQ(\gamma) = C\overline{A}^* SQ(\gamma) \\ \iff C\overline{A}^* BU(\gamma) \preceq C\overline{A}^* SQ(\gamma). \end{aligned} \tag{6}$$

According to Definition 3, solving the DDP in event-domain means that the control $U(\gamma)$ has to achieve

$$\begin{aligned} C\overline{A}^* BU(\gamma) \oplus C\overline{A}^* SQ(\gamma) = C\overline{A}^* BU(\gamma) \\ \iff C\overline{A}^* SQ(\gamma) \preceq C\overline{A}^* BU(\gamma). \end{aligned} \tag{7}$$

In this paper, all disturbances are assumed to be measurable, so if $U(\gamma) = V(\gamma) = PQ(\gamma)$, by considering the residuation

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theory(see [23]) and Eq. (6), the MDDP is solved if and only if the inequality

$$C\overline{A}^*BU(\gamma) = C\overline{A}^*BPQ(\gamma) \preceq C\overline{A}^*SQ(\gamma)$$

holds and the greatest solution solving this inequality is

$$\begin{aligned} P_{opt} &= \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*S\right) \qquad (8)\\ &= \bigoplus_{P\in\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^{p\times r}} \{C\overline{A}^*BP \preceq C\overline{A}^*S\}, \end{aligned}$$

i.e. such a P_{opt} solves the MDDP for any disturbance $Q(\gamma)$.

Theorem 1 ([23]): The optimal pre-filter $V(\gamma) = P_{opt}Q(\gamma)$, which solves the MDDP, also solves the DDP for the max-plus linear systems described in Eq. (4) if and only if $\mathsf{Im}\ C\overline{A}^*S \subset \mathsf{Im}\ C\overline{A}^*B$, i.e. $C\overline{A}^*S = C\overline{A}^*B((C\overline{A}^*B)\circ\!\!\!\backslash(C\overline{A}^*S))$.

C. Solving DDP and MDDP by a State-Feedback Control

If we want to find a state-feedback control $u_K(k) = Kx(k-1) \oplus v(k)$ to solve DDP and MDDP, then we can represent the control in γ-domain as $U_{\overline{K}}(\gamma) = \overline{K}X(\gamma) \oplus V(\gamma)$, where $\overline{K} = \gamma K$. Moreover, such a $\overline{K}$ can be generalized to entries that are periodic series with γ^d, $d \geq 1$.

Mathematically, the state and output signals in the event γ-domain are represented as follows:

$$\begin{aligned} X(\gamma) &= (\overline{A}\oplus B\overline{K})^*BV(\gamma) \oplus (\overline{A}\oplus B\overline{K})^*SQ(\gamma)\\ &= (\overline{A}\oplus B\overline{K})^*\,[B \mid S] \begin{pmatrix} V(\gamma)\\ Q(\gamma)\end{pmatrix}\\ &= (\overline{A}\oplus B\overline{K})^*\widetilde{B} \begin{pmatrix} V(\gamma)\\ Q(\gamma)\end{pmatrix}, \text{ where } \widetilde{B} = [B \mid S],\\ Y(\gamma) &= CX(\gamma) = C(\overline{A}\oplus B\overline{K})^*\widetilde{B} \begin{pmatrix} V(\gamma)\\ Q(\gamma)\end{pmatrix}. \qquad (9) \end{aligned}$$

Based on Definition 3, solving the DDP in event-domain means that the state-feedback controller has to achieve the following equality:

$$C(\overline{A}\oplus B\overline{K})^*\widetilde{B} \begin{pmatrix} V(\gamma)\\ Q(\gamma)\end{pmatrix} = C(\overline{A}\oplus B\overline{K})^*BV(\gamma). \qquad (10)$$

Based on Definition 4, solving the MDDP in event-domain means that the state-feedback controller has to achieve another equality:

$$C(\overline{A}\oplus B\overline{K})^*\widetilde{B} \begin{pmatrix} V(\gamma)\\ Q(\gamma)\end{pmatrix} = C(\overline{A}\oplus B\overline{K})^*SQ(\gamma). \qquad (11)$$

Equations (10) and (11) each have three variables, the state-feedback structure $\overline{K}$, the open-loop controller $V(\gamma)$, as well as the disturbance input $Q(\gamma)$.

For an open-loop control $V(\gamma)$, the goal is to find a state-feedback controller $\overline{K}$ in equations (10) and (11) such that the output signals generated by the state-feedback control $U_{\overline{K}}(\gamma) = \overline{K}X(\gamma) \oplus V(\gamma)$ are the same as the output signals generated by the open-loop control $V(\gamma)$. In summary, that is, the following equality holds

$$C(\overline{A}\oplus B\,\overline{K})^*\widetilde{B} \begin{pmatrix} V(\gamma)\\ Q(\gamma)\end{pmatrix} = C\overline{A}^*\widetilde{B} \begin{pmatrix} V(\gamma)\\ Q(\gamma)\end{pmatrix}. \qquad (12)$$

Proposition 1: ([14], [16]) The greatest controller $\overline{K}_{opt}$ is given by

$$\overline{K}_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*\widetilde{B}\right) \circ\!\!\!/ \left(\overline{A}^*\widetilde{B}\right), \qquad (13)$$

such that the output trajectories generated by the state feedback controller are the same as the output trajectories generated by the open-loop controller, i.e. the equality $C\left(\overline{A}\oplus B\,\overline{K}_{opt}\right)^*\widetilde{B} = C\overline{A}^*\widetilde{B}$ holds.

Proposition 2: ([24]) The state-feedback control law $U_{\overline{K}_{opt}}(\gamma) = \overline{K}_{opt}X(\gamma) \oplus P_{opt}Q(\gamma)$ solves the MDDP of the max-plus linear system in Eq. (4), where $P_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*S\right)$ and $\overline{K}_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*\widetilde{B}\right) \circ\!\!\!/ \left(\overline{A}^*\widetilde{B}\right)$.

Proposition 3: ([24]) The state-feedback control law $U_{\overline{K}_{opt}}(\gamma) = \overline{K}_{opt}X(\gamma) \oplus P_{opt}Q(\gamma)$, where $P_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*S\right)$ and $\overline{K}_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*\widetilde{B}\right) \circ\!\!\!/ \left(\overline{A}^*\widetilde{B}\right)$, solves the DDP of the max-plus linear system in Eq. (4) if and only if $\mathsf{Im}\ C\overline{A}^*S \subset \mathsf{Im}\ C\overline{A}^*B$.

D. Solving DDP and MDDP by an Output Feedback Control

If we want to find an output feedback control $u_F(k) = Fy(k) \oplus v(k)$, i.e. $u_F(k) = FCx(k) \oplus v(k)$ to solve DDP and MDDP, then we can represent the control in γ-domain as $U_F(\gamma) = FY(\gamma) \oplus V(\gamma) = FCX(\gamma) \oplus V(\gamma)$. Similar steps as the previous subsection on state-feedback control can be taken by simply replacing $\overline{K}$ by FC.

Proposition 4: ([24]) The greatest controller F_{opt} is given by

$$F_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*\widetilde{B}\right) \circ\!\!\!/ \left(C\overline{A}^*\widetilde{B}\right), \qquad (14)$$

such that the output trajectories generated by the output feedback control are the same as the output trajectories generated by the open-loop controller, i.e. the equality $C\left(\overline{A}\oplus B\,F_{opt}C\right)^*\widetilde{B} = C\overline{A}^*\widetilde{B}$ holds.

Proposition 5: ([24]) The output feedback control law $U_{F_{opt}}(\gamma) = F_{opt}Y(\gamma) \oplus P_{opt}Q(\gamma)$ solves the MDDP of the max-plus linear system in Eq. (4), where $P_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*S\right)$ and $F_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*\widetilde{B}\right) \circ\!\!\!/ \left(C\overline{A}^*\widetilde{B}\right)$.

Proposition 6: ([24]) The output feedback control law $U_{F_{opt}}(\gamma) = F_{opt}Y(\gamma) \oplus P_{opt}Q(\gamma)$, where $P_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*S\right)$ and $F_{opt} = \left(C\overline{A}^*B\right) \circ\!\!\!\backslash \left(C\overline{A}^*\widetilde{B}\right) \circ\!\!\!/ \left(C\overline{A}^*\widetilde{B}\right)$, solves the DDP of the max-plus linear system in Eq. (4) if and only if $\mathsf{Im}\ C\overline{A}^*S \subset \mathsf{Im}\ C\overline{A}^*B$.

In summary, the open-loop controller, state-feedback controller, and the output feedback controller can be illustrated in Fig. 1 in solid lines, dashed lines, and dotted lines, respectively. If $P = P_{opt}$, $\overline{K} = \overline{K}_{opt}$, and $F = F_{opt}$, then the open-loop controller, state-feedback controller, and the output feedback controller can solve the MDDP, respectively. Moreover, based on Theorem 1, Proposition 3 and Proposition 6, if the image inclusion condition $\mathsf{Im}\ C\overline{A}^*S \subset \mathsf{Im}\ C\overline{A}^*B$ is satisfied, these three controllers can solve DDP as well.

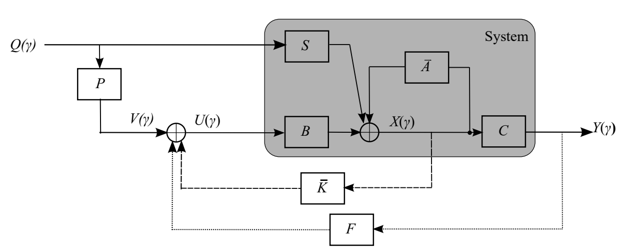

Fig. 1: The controller structure for DDP and MDDP.

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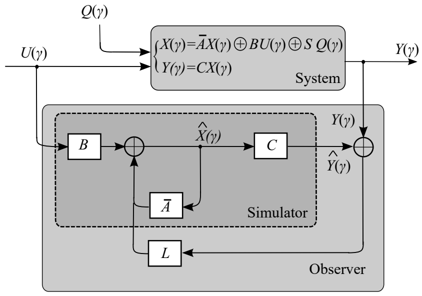

Fig. 2: The observer structure of max-plus linear systems.

V. Max-plus Observer

Fig. 2 depicts the observer structure directly inspired by the Luenberger observer in classical linear system theory ([10],[11],[17]). The observer matrix L is used to provide information from the system output into the simulator in order to take the disturbances $Q(\gamma)$ acting on the system into account. The simulator is described by the model[2](matrices $\overline{A}$, B, C) which is assumed to represent the fastest behavior of the real system in a guaranteed way[3], furthermore, the simulator is initialized by the canonical initial conditions ($i.e.,\ \ \hat{x}_i(k)=\varepsilon, \forall k\leq 0$).

By considering the configuration in Fig. 2 and these assumptions, the computation of the optimal observer matrix L will be proposed in order to achieve the constraint $\hat{X}(\gamma)\preceq X(\gamma)$. Optimality means that matrix L is the greatest one, according to the residuation theory. Therefore, the estimated state $\hat{X}(\gamma)$ is the greatest which achieves the objective, in other words, it is as close as possible to $X(\gamma)$. As in the development proposed in conventional linear systems theory, matrices A, B, C and R are assumed to be known, then the system trajectories are given by Eq. (5). According to Fig. 2, the observer equations, similarly as the Luenberger observer, are given by:

$$\begin{aligned}\hat{X}(\gamma) &= \overline{A}\hat{X}(\gamma)\oplus BU(\gamma)\oplus L(\hat{Y}(\gamma)\oplus Y(\gamma))\\ &= \overline{A}\hat{X}(\gamma)\oplus BU(\gamma)\oplus LC\hat{X}(\gamma)\oplus LCX(\gamma)\\ &= (\overline{A}\oplus LC)\hat{X}(\gamma)\oplus BU(\gamma)\oplus LCX(\gamma),\\ &= (\overline{A}\oplus LC)^*BU(\gamma)\oplus(\overline{A}\oplus LC)^*LCX(\gamma)\\ &= (\overline{A}\oplus LC)^*BU(\gamma)\oplus\\ &\quad (\overline{A}\oplus LC)^*LC(\overline{A}^*BU(\gamma)\oplus\overline{A}^*SQ(\gamma)) \qquad (15)\end{aligned}$$

By considering Eq. (f.1) in Appendix, the following equality is obtained:

$$(\overline{A}\oplus LC)^*=\overline{A}^*(LC\overline{A}^*)^*, \qquad (16)$$

by replacing in Eq. (15):

$$\begin{aligned}\hat{X}(\gamma) &= \overline{A}^*(LC\overline{A}^*)^*BU(\gamma)\oplus\\ &\quad \overline{A}^*(LC\overline{A}^*)^*LC\overline{A}^*BU(\gamma)\oplus\\ &\quad \overline{A}^*(LC\overline{A}^*)^*LC\overline{A}^*SQ(\gamma),\end{aligned}$$

and by denoting $(LC\overline{A}^*)^*LC\overline{A}^*=(LC\overline{A}^*)^+$, this equation may be written as follows:

$$\begin{aligned}\hat{X}(\gamma) &= \overline{A}^*(LC\overline{A}^*)^*BU(\gamma)\oplus\overline{A}^*(LCA^*)^+BU(\gamma)\\ &\quad \oplus\overline{A}^*(LC\overline{A}^*)^+SQ(\gamma).\end{aligned}$$

[2]Disturbances are uncontrollable and *a priori* unknown, then the simulator does not take them into account.

[3]Unlike in the conventional linear system theory, this assumption means that the fastest behavior of the system is assumed to be known and that the disturbances can only delay its behavior.

Since $(LC\overline{A}^*)^*\succeq(LC\overline{A}^*)^+=(LC\overline{A}^*)^*LC\overline{A}^*$, the observer model may be written as follows:

$$\begin{aligned}\hat{X}(\gamma) &= \overline{A}^*(LC\overline{A}^*)^*BU(\gamma)\oplus\overline{A}^*(LC\overline{A}^*)^+SQ(\gamma)\\ &= (\overline{A}\oplus LC)^*BU(\gamma)\oplus(\overline{A}\oplus LC)^*LC\overline{A}^*SQ(\gamma), \qquad (17)\\ &\quad \text{due to Eq.(16).}\end{aligned}$$

As said previously, the objective considered is to compute the greatest observation matrix L, denoted as L_{opt}, such that the estimated state vector $\hat{X}(\gamma)$ is as close as possible to state x, under the constraint $\hat{X}(\gamma)\preceq X(\gamma)$, formally it can be written as, finding the greatest L satisfying the following inequality, $\forall U(\gamma),\ Q(\gamma)$:

$$\begin{aligned}\hat{X}(\gamma) &= (\overline{A}\oplus LC)^*BU(\gamma)\oplus(\overline{A}\oplus LC)^*LC\overline{A}^*SQ(\gamma)\\ &\preceq X(\gamma)=\overline{A}^*BU(\gamma)\oplus\overline{A}^*SQ(\gamma), \qquad (18)\end{aligned}$$

or equivalently:

$$\begin{aligned}&(\overline{A}\oplus LC)^*B\preceq\overline{A}^*B,\\ &(\overline{A}\oplus LC)^*LC\overline{A}^*S\preceq\overline{A}^*S.\end{aligned}$$

Lemma 1 ([11]): The following equivalence holds:

$$(\overline{A}\oplus LC)^*B=\overline{A}^*B\Leftrightarrow L\preceq L_1=(\overline{A}^*B)\not\phi(C\overline{A}^*B).$$

Lemma 2 ([11]): The following equivalence holds:

$$(\overline{A}\oplus LC)^*LC\overline{A}^*S\preceq\overline{A}^*S\Leftrightarrow L\preceq L_2=(\overline{A}^*S)\not\phi(C\overline{A}^*S).$$

Proposition 7 ([11]): $L_{opt}\ =\ L_1\wedge L_2$ is the greatest observer matrix L such that: $\hat{X}(\gamma)\preceq X(\gamma)\quad\forall(U(\gamma),Q(\gamma))$.

Remark 1: The canonical injection from the causal elements of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ (denoted $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]^+$) in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ is also residuated (see [7] for details). Its residual is given by $\mathsf{Pr}\left(\bigoplus_{k\in\mathbb{Z}}s(k)\gamma^k\right)=\bigoplus_{k\in\mathbb{Z}}s_+(k)\gamma^k$ where

$$s_+(k)=\begin{cases}s(k) & \text{if } (k,s(k))\geq(0,0),\\ \varepsilon & \text{otherwise.}\end{cases}$$

Notice that the preceding lemmas and proposition could be restricted to causal projections $L_{opt_+}=\mathsf{Pr}_+(L_{opt})$, which is the greatest causal solution for $(\overline{A}\oplus LC)^*B\preceq\overline{A}^*B$ and $(\overline{A}\oplus LC)^*LC\overline{A}^*R\preceq\overline{A}^*S$, with $\hat{X}(\gamma)\preceq X(\gamma)$.

Corollary 1 ([12]): Matrix $L_{opt_+}\ =\ \mathsf{Pr}_+(L_{opt})$ is the greatest causal observer ensuring the equality between the estimated output $Y(\hat{\gamma})$ and the measured output $Y(\gamma)$.

Remark 2: If the observer matrix L becomes L_{opt+}, then the max-plus observer in Eq. (15) will become

$$\hat{X}(\gamma)=\overline{A}\hat{X}(\gamma)\oplus BU(\gamma)\oplus L_{opt+}Y(\gamma), \qquad (19)$$

because $Y(\gamma)$ is the same as $\hat{Y}(\gamma)$.

VI. Observer-Based Controller Solving DDP and MDDP

As in the classical theory, the state often is not measurable or it is too expensive to measure all the states. Hence, in this section, we propose to use the estimated state, obtained thanks to the observer proposed in the previous section, to compute the observer-based state-feedback control law. Then, this control strategy is compared with the output feedback control as given in Proposition 5. Formally, the observer-based control $u_M(k)=M\hat{x}(k-1)\oplus v(k)$, or in γ-domain as $U_{\overline{M}}(\gamma)=\overline{M}\hat{X}(\gamma)\oplus V(\gamma)$, where $\overline{M}=\gamma M$ and

$$\begin{aligned}&\hat{X}(\gamma)=\overline{A}\hat{X}(\gamma)\oplus BU_M(\gamma)\oplus L_{opt}(CX(\gamma)\oplus C\hat{X}(\gamma))\\ &=(\overline{A}\oplus L_{opt}C)^*BU_M(\gamma)\oplus(\overline{A}\oplus L_{opt}C)^*L_{opt}C\overline{A}^*SQ(\gamma)\\ &\qquad\text{(See Eq.(17)).}\end{aligned}$$

The optimal observer matrix $L_{opt}=L_1\wedge L_2$ is the same as introduced in Proposition 7, where $L_1=(A^*B)\not\phi(CA^*B)$ and $L_2=(A^*S)\not\phi(CA^*S)$. Moreover, this optimal observer matrix is clearly independent of the control law u_M.

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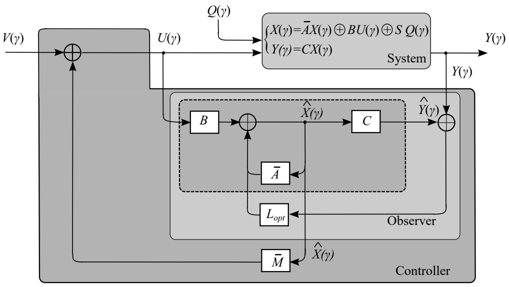

Fig. 3: The observer-based controller for max-plus linear systems.

This observer-based control strategy in γ-domain is depicted in Fig. 3 and can be rewritten as

$$\begin{aligned}&U_{\overline{M}}(\gamma) = \overline{M}\hat{X}(\gamma) \oplus V(\gamma)\\ &= \overline{M}(\overline{A} \oplus L_{opt}C)^* BU_{\overline{M}}(\gamma)\\ &\oplus \overline{M}(\overline{A} \oplus L_{opt}C)^* L_{opt}C\overline{A}^* SQ(\gamma) \oplus V(\gamma)\\ &= (\overline{M}(\overline{A} \oplus L_{opt}C)^* B)^* V(\gamma) \oplus \qquad (20)\\ &(\overline{M}(\overline{A} \oplus L_{opt}C)^* B)^* \overline{M}(\overline{A} \oplus L_{opt}C)^* L_{opt}C\overline{A}^* SQ(\gamma)\end{aligned}$$

Therefore, plugging in $U(\gamma)$ above to the system state, we can obtain as

$$\begin{aligned}&X(\gamma) = \overline{A}^* BU_{\overline{M}}(\gamma) \oplus \overline{A}^* SQ(\gamma)\\ &= \overline{A}^* B(\overline{M}(\overline{A} \oplus L_{opt}C)^* B)^* V(\gamma) \oplus\\ &\overline{A}^* B(\overline{M}((\overline{A} \oplus L_{opt}C)^* B)^* \overline{M}(\overline{A} \oplus L_{opt}C)^* L_{opt}C\overline{A}^* SQ(\gamma)\\ &\oplus \overline{A}^* SQ(\gamma),\end{aligned}$$

and then the system output can be rewritten accordingly as

$$\begin{aligned}&Y(\gamma) = CA^* BU_{\overline{M}}(\gamma) \oplus CA^* SQ(\gamma)\\ &= C\overline{A}^* B(\overline{M}(\overline{A} \oplus L_{opt}C)^* B)^* V(\gamma) \oplus\\ &C\overline{A}^* B(\overline{M}((\overline{A} \oplus L_{opt}C)^* B)^* \overline{M}(\overline{A} \oplus L_{opt}C)^* L_{opt}C\overline{A}^* SQ(\gamma)\\ &\oplus C\overline{A}^* SQ(\gamma), \qquad (21)\end{aligned}$$

Proposition 8: The greatest observer-based controller $\overline{M}_{opt}$ is given by

$$\overline{M}_{opt} = \left(C\overline{A}^* B\right) \lmoustache \left(C\overline{A}^* \widetilde{B}\right) \rmoustache \left(\overline{A}^* \widetilde{B}\right) = \overline{K}_{opt}, \qquad (22)$$

where $\widetilde{B} = [B \mid S]$, such that the output trajectories generated by the observer-based control $U_{\overline{M}}(\gamma) = \overline{M}\hat{X}(\gamma) \oplus V(\gamma)$, where $\hat{X}(\gamma) = \overline{A}\hat{X}(\gamma) \oplus BU_{\overline{M}}(\gamma) \oplus L_{opt}(\hat{Y}(\gamma) \oplus Y(\gamma))$, are the same as the output trajectories generated by the open-loop control $U(\gamma) = V(\gamma)$.

Proof: In order to prove that the output trajectories generated by the observer based controller $\overline{M}$ as described in Eq. (21) preserve the same output trajectories as the open-loop control in Eq. (5), we need to prove

$$C\overline{A}^* B(\overline{M}(\overline{A} \oplus L_{opt}C)^* B)^* = C\overline{A}^* B, \qquad (23)$$

and

$$\begin{aligned}&C\overline{A}^* B(\overline{M}((\overline{A} \oplus L_{opt}C)^* B))^* \overline{M}(\overline{A} \oplus L_{opt}C)^* L_{opt}C\overline{A}^* S\\ &= C\overline{A}^* S, \qquad (24)\end{aligned}$$

respectively. For Eq. (23), clearly $C\overline{A}^* B(\overline{M}(\overline{A} \oplus L_{opt}C)^* B)^* \succeq C\overline{A}^* B$ due to the definition of the star operation. The other direction can be proved as follows: the left hand side of the equality can be written as

$$C\overline{A}^* B(\overline{M}(\overline{A} \oplus L_{opt}C)^* B)^* = C\overline{A}^* B(\overline{M}\overline{A}^* B)^*,$$

due to Lemma 1. Moreover, to ensure the inequality $C\overline{A}^* B(\overline{M}(\overline{A} \oplus L_{opt}C)^* B)^* \preceq C\overline{A}^* B$, we need to have

$$\begin{aligned}C\overline{A}^* B(\overline{M}\overline{A}^* B)^* &\preceq C\overline{A}^* B\\ \Leftrightarrow (\overline{M}\overline{A}^* B)^* &\preceq (C\overline{A}^* B) \lmoustache (C\overline{A}^* B)\\ \Leftrightarrow (\overline{M}\overline{A}^* B)^* &\preceq ((C\overline{A}^* B) \lmoustache (C\overline{A}^* B))^* \text{ due to Eq.(2)}\\ \Leftrightarrow \overline{M}\overline{A}^* B &\preceq (C\overline{A}^* B) \lmoustache (C\overline{A}^* B)\\ \Leftrightarrow \overline{M} &\preceq (C\overline{A}^* B) \lmoustache (C\overline{A}^* B) \rmoustache (\overline{A}^* B). \qquad (25)\end{aligned}$$

Therefore, for any $M \preceq (C\overline{A}^* B) \lmoustache (C\overline{A}^* B) \rmoustache (\overline{A}^* B)$, Eq. (23) is satisfied.

To prove that Eq. (24) also holds, assuming that $M \preceq (C\overline{A}^* B) \lmoustache (C\overline{A}^* B) \rmoustache (\overline{A}^* B)$, i.e. Eq. (23) is already satisfied, then the left hand side of the equality in Eq. (24) can be written as

$$\begin{aligned}&C\overline{A}^* B(\overline{M}((\overline{A} \oplus L_{opt}C)^* B)^* \overline{M}(A \oplus L_{opt}C)^* L_{opt}C\overline{A}^* S\\ =\ &C\overline{A}^* B\overline{M}(\overline{A} \oplus L_{opt}C)^* L_{opt}C\overline{A}^* S, \text{ due to Eq.(23)}\\ =\ &C\overline{A}^* B\overline{M}\overline{A}^* S, \text{ due to Lemma 2.}\end{aligned}$$

In order to have Eq. (24) holds, we have

$$\begin{aligned}&C\overline{A}^* B\overline{M}\overline{A}^* S \preceq (C\overline{A}^* B)\\ \Leftrightarrow\ &\overline{M}\overline{A}^* S \preceq (C\overline{A}^* B) \lmoustache (C\overline{A}^* S)\\ \Leftrightarrow\ &\overline{M} \preceq (C\overline{A}^* B) \lmoustache (C\overline{A}^* S) \rmoustache (\overline{A}^* S). \qquad (26)\end{aligned}$$

Hence, if $\overline{M}$ satisfies both inequalities Eq. (25) and Eq. (26), equivalently, Eq. (23) and Eq. (24) are both satisfied. Moreover, the following inequalities hold as well due to residuation theory:

$$\begin{aligned}&C\overline{A}^* B\overline{M}\overline{A}^* B \preceq C\overline{A}^* B, \text{ and } C\overline{A}^* B\overline{M}\overline{A}^* S \preceq C\overline{A}^* S\\ &\qquad \Leftrightarrow C\overline{A}^* B\overline{M}\overline{A}^* \widetilde{B} \preceq C\overline{A}^* \widetilde{B} \Leftrightarrow\\ &\overline{M} \preceq \overline{M}_{opt} = \left(C\overline{A}^* B\right) \lmoustache \left(C\overline{A}^* \widetilde{B}\right) \rmoustache \left(\overline{A}^* \widetilde{B}\right) = \overline{K}_{opt},\end{aligned}$$

where $\widetilde{B} = [B \mid S]$, i.e. the observer-based controller $\overline{M}_{opt}$ is the same as the state-feedback controller $\overline{K}_{opt}$ in Proposition 1. ■

Thanks to the Separation Principle, Proposition 8 shows that the controller synthesis and the observer synthesis can be obtained independently. In another words, first, we can find the greatest observer matrix L_{opt} to ensure that the estimated output is the same as the original output. Second, we can find the greatest state-feedback matrix $\overline{M}_{opt}$, i.e. $\overline{K}_{opt}$ to ensure that MDDP is solved. After combining the greatest observer matrix L_{opt} and the state-feedback matrix $\overline{K}_{opt}$, the observer-based controller is constructed and denoted as $U_{\overline{M}_{opt}}(\gamma) = \overline{M}_{opt}\hat{X}(\gamma) \oplus V(\gamma)$, where $\hat{X}(\gamma) = \overline{A}\hat{X}(\gamma) \oplus BU_{\overline{M}_{opt}}(\gamma) \oplus L_{opt}(\hat{Y}(\gamma) \oplus Y(\gamma))$.

Proposition 9: The observer-based control law $U_{\overline{M}_{opt}}(\gamma) = \overline{M}_{opt}\hat{X}(\gamma) \oplus P_{opt}Q(\gamma)$ solves the MDDP of the max-plus linear system in Eq. (4), where $P_{opt} = \left(C\overline{A}^* B\right) \lmoustache \left(C\overline{A}^* S\right)$ and $\overline{M}_{opt} = \left(C\overline{A}^* B\right) \lmoustache \left(C\overline{A}^* \widetilde{B}\right) \rmoustache \left(\overline{A}^* \widetilde{B}\right)$.

Proof: If $\overline{M}_{opt}$ and the observer matrix L_{opt} are able to preserve the output trajectories the same as the open-loop trajectories $C\overline{A}^* BV(\gamma) \oplus C\overline{A}^* SQ(\gamma)$. The open-loop control $V(\gamma) = P_{opt}Q(\gamma)$ produces the outputs $C\overline{A}^* BP_{opt}Q(\gamma) \preceq C\overline{A}^* SQ(\gamma)$, therefore, the MDDP is solved. ■

Proposition 10: The observer-based control law $U_{\overline{M}_{opt}}(\gamma) = \overline{M}_{opt}\hat{X}(\gamma) \oplus P_{opt}Q(\gamma)$, where $P_{opt} = \left(C\overline{A}^* B\right) \lmoustache \left(C\overline{A}^* S\right)$ and $\overline{M}_{opt} = \left(C\overline{A}^* B\right) \lmoustache \left(C\overline{A}^* \widetilde{B}\right) \rmoustache \left(\overline{A}^* \widetilde{B}\right)$, solves the DDP of the max-plus linear system in Eq. (4) if and only if $\mathrm{Im}\, C\overline{A}^* S \subset \mathrm{Im}\, C\overline{A}^* B$.

Proof: The observer-based control law $U_{\overline{M}_{opt}}(\gamma) = \overline{M}_{opt}\hat{X}(\gamma) \oplus P_{opt}Q(\gamma)$ solves MDDP. If the image inclusion condition is satisfied, $\mathrm{Im}\, C\overline{A}^* S \subset \mathrm{Im}\, C\overline{A}^* B$, then, due to Theorem 1, the DDP is solve as well. ■

The observer-based controller solving the MDDP and DDP is illustrated in Fig. 4.

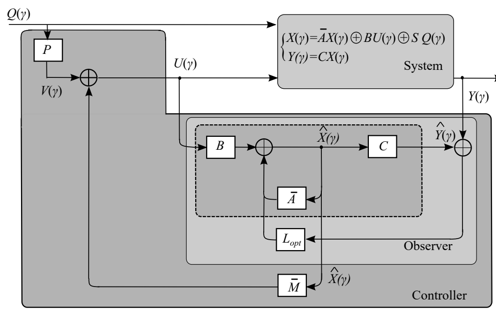

Fig. 4: The observer-based controller for max-plus linear systems.

Proposition 11: The output feedback control law $U_{F_{opt}}(\gamma) = F_{opt}Y(\gamma) \oplus V(\gamma)$, the observer-based control law $U_{\overline{M}_{opt}}(\gamma) = \overline{M}_{opt}\hat{X}(\gamma) \oplus V(\gamma)$, and the state-feedback control law $U_{\overline{K}_{opt}}(\gamma) = \overline{K}_{opt}X(\gamma) \oplus V(\gamma)$ are ordered as follows:

$$U_{F_{opt}}(\gamma) \preceq U_{\overline{M}_{opt}}(\gamma) \preceq U_{\overline{K}_{opt}}(\gamma).$$

Proof: According to Equality f.(19) in Appendix, the following equality holds: $F_{opt} = \overline{M}_{opt} \phi C$. Hence, $F_{opt}C \preceq \overline{M}_{opt} \Rightarrow F_{opt}C\hat{X}(\gamma) \preceq \overline{M}_{opt}\hat{X}(\gamma)$. According to Corollary 1, $C\hat{X}(\gamma) = \hat{Y}(\gamma) = Y(\gamma)$, hence $F_{opt}Y(\gamma) \preceq \overline{M}_{opt}\hat{X}(\gamma)$. This means that the feedback control taking the output into account is smaller than or equal to the observer-based control using the estimated state. By recalling that the addition and product laws are order preserving, it appears that:

$$U_{F_{opt}}(\gamma) = F_{opt}Y(\gamma) \oplus V(\gamma) \preceq U_{\overline{M}_{opt}}(\gamma) = \overline{M}_{opt}\hat{X}(\gamma) \oplus V(\gamma).$$

According to Proposition 7, $\hat{X}(\gamma) \preceq X(\gamma)$, and, according to Proposition 8, $\overline{M}_{opt} = \overline{M}_{opt}$. Hence, $U_{\overline{M}_{opt}}(\gamma) \preceq U_{\overline{K}_{opt}}(\gamma)$. ■

According to the just-in-time criterion, Proposition 11 shows that the observer-based control strategy is better than the output feedback control strategy. For instance, in a manufacturing setting, the observer-based control would provide a better scheduling by starting the process later than the output feedback control, while ensuring the same output parts finishing time. This scheduling would allow users to load the raw parts later rather than earlier to avoid unnecessary congestions in the manufacturing lines.

VII. Application to a High Throughput Screening System in Drug Discovery

High throughput screening (HTS) is a standard technology in drug discovery. In HTS systems, optimal scheduling is required to finish the screening in the shortest time, as well as to preserve the consistent time spent on each activity. This section is using a HTS system to illustrate the main results in this paper. This HTS system, adapted from [4], has three nested activities running on three different single-capacity resources: pipettor (activity 1), reader (activity 2), and incubator (activity 3). The Gantt chart for this HTS system is shown in Fig. 5. One cycle of events is shown as follows: first, the pipettor drops the DNA/RNA compounds into the microplate, then the microplate is transferred to the reader to be scanned, and then the microplate is transferred to the incubator. After the first cycle of events, the second cycle of event will start. Moreover, the three activities are overlapping during the transition time, for instance, the reader starts scanning 3 time units before the pipettor finishes its task, and finishes scanning 7 units after incubator starts the task, as shown in Fig. 5.

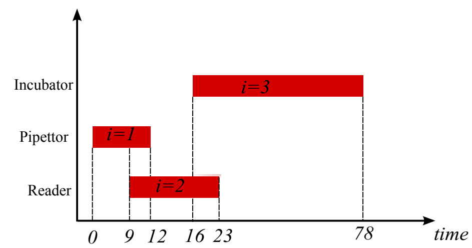

Fig. 5: The Gantt chart of one cycle of activities.

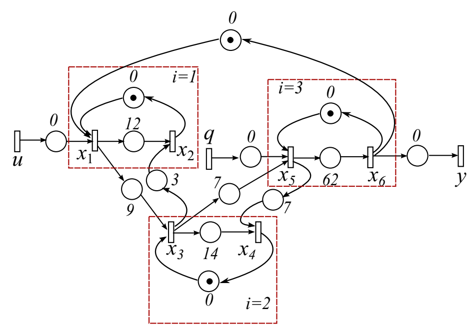

Fig. 6: The TEG model for the HTS system.

If we are interested in the start and release event time of each activity, we can model the HTS system as a TEG model, shown in Fig. 6, in which x_1 and x_2 denote the start and release time of the activity 1 on the pipettor, x_3 and x_4 denote the start and release time of the activity 2 on the reader, and x_5 and x_6 denote the start and release time of the activity 3 on the incubator. The input u is the starting time of the pipettor which users can decide: when to load the chemical compounds. The disturbance q is the starting time of the incubator, such as transition time delay from the reader to the incubator due to system malfunction. The output y is the release time of the incubator. The cycles indicate places and the bars represent the transitions x_i. The tokens in the places represent that the transitions are ready to be fired, i.e. the activity is ready to start. For the TEG model of a HTS system shown in Fig. 6, the system over the max-plus algebra $\mathbb{Z}_{\max}[\![\gamma]\!]$ is described as the following:

$$\begin{aligned}
X(\gamma) &= \overline{A}X(\gamma) \oplus BU(\gamma) \oplus SQ(\gamma) \\
Y(\gamma) &= CX(\gamma), \text{ where} \\
\overline{A} &= \begin{bmatrix} \varepsilon & \gamma & \varepsilon & \varepsilon & \varepsilon & \gamma \\ 12 & \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon \\ 9 & \varepsilon & \varepsilon & \gamma & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 14 & \varepsilon & 7 & \varepsilon \\ \varepsilon & \varepsilon & 7 & \varepsilon & \varepsilon & \gamma \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 62 & \varepsilon \end{bmatrix}, \\
B^T &= [\, e \;\; \varepsilon \;\; \varepsilon \;\; \varepsilon \;\; \varepsilon \;\; \varepsilon \,], \\
S^T &= [\, \varepsilon \;\; \varepsilon \;\; \varepsilon \;\; \varepsilon \;\; e \;\; \varepsilon \,], \text{ and} \\
C &= [\, \varepsilon \;\; \varepsilon \;\; \varepsilon \;\; \varepsilon \;\; \varepsilon \;\; e \,].
\end{aligned}$$

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The example has been computed by using toolbox `MinMaxGD`, a C++ library which allows the handling of periodic series as introduced in ([6]), and it can be noted that this library is also interfaced with Scilab and MATLAB. We obtain The transfer function between the output $Y(\gamma)$ and disturbance $U(\gamma)$ and the input $Q(\gamma)$, respectively, as

$$\begin{aligned} C\overline{A}^*B &= 78(78\gamma)^* = 78 \oplus 156\gamma \oplus 234\gamma^2 \cdots, \\ C\overline{A}^*S &= 62(78\gamma)^* = 62 \oplus 140\gamma \oplus 218\gamma^2 \cdots, \end{aligned}$$

in which each component of these matrices is a periodic series. Essentially, the γ-periodic series represent the output sequence when an infinite number of supplies are put in the system at time 0 (impulse input). For instance, $C\overline{A}^*B$ represents the impulse response of the incubator as $y(0) = 78$, $y(1) = 156$, $y(2) = 234$, etc.

The non-causal filter P_{opt} is $\left(C\overline{A}^*B\right) \circ\!\!\!\!\backslash \left(C\overline{A}^*S\right) = -16(78\gamma)^*$. In this example, we have the image inclusion condition $\mathsf{Im}\ C\overline{A}^*S \subset \mathsf{Im}\ C\overline{A}^*B$ satisfied, hence, this non-causal prefilter solves the MDDP and the DDP at the same time due to $C\overline{A}^*BP_{opt} = C\overline{A}^*S$. This prefilter P_{opt} is not causal because there are negative coefficients in the matrix. If we take the canonical injection from the causal elements of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, then the greatest causal prefilter is

$$P_{opt+} = \mathsf{Pr}(P_{opt}) = 62\gamma(78\gamma)^*.$$

The causal filter P_{opt+} for the MDDP does not solve DDP because

$$C\overline{A}^*BP_{opt+} = 140\gamma(78\gamma)^* \prec C\overline{A}^*S.$$

Next, we construct the observer-based control $U_{\overline{M}_{opt}}(\gamma) = \overline{M}_{opt}\hat{X}(\gamma) \oplus P_{opt}Q(\gamma)$ with $\hat{X} = A\hat{X}(\gamma) \oplus BU_{\overline{M}_{opt}}(\gamma) \oplus L_{opt}(Y(\gamma) \oplus \hat{Y}(\gamma))$. According to Lemma 1, Lemma 2, the observer matrix L_{opt} is given as follows:

$$\begin{aligned} L_{opt} &= L_1 \wedge L_2 = (A^*B) \circ\!\!\!\!/ (CA^*B) \wedge (A^*R) \circ\!\!\!\!/ (CA^*R) \\ &= \begin{bmatrix} \gamma(78\gamma)^* \\ 12\gamma(78\gamma)^* \\ 9\gamma(78\gamma)^* \\ -55(78\gamma)^* \\ -62(78\gamma)^* \\ (78\gamma)^* \end{bmatrix}, \end{aligned}$$

Then, the causal observer matrix L_{opt+} is

$$L_{opt+} = \begin{bmatrix} \gamma(78\gamma)^* \\ 12\gamma(78\gamma)^* \\ 9\gamma(78\gamma)^* \\ 23\gamma(78\gamma)^* \\ 16\gamma(78\gamma)^* \\ (78\gamma)^* \end{bmatrix}.$$

According to Proposition 8 and Eq. (22), the greatest observer-based controller preserving the open-loop behaviors $\overline{M}_{opt}$ is obtained as follows:

$$\begin{aligned} \overline{M}_{opt} &= \left(C\overline{A}^*B\right) \circ\!\!\!\!\backslash \left(C\overline{A}^*\widetilde{B}\right) \circ\!\!\!\!/ \left(\overline{A}^*\widetilde{B}\right) \\ &= [(78\gamma)^*, -12(78\gamma)^*, -9(78\gamma)^*, \\ &\quad -23(78\gamma)^*, -16(78\gamma)^*, -78(78\gamma)^*]. \end{aligned}$$

The greatest causal feedback is

$$\begin{aligned} \overline{M}_{opt+} &= \mathsf{Pr}(\overline{M}_{opt}) = [(78\gamma)^*, 66\gamma(78\gamma)^*, 69\gamma(78\gamma)^*, \\ &\quad 55\gamma(78\gamma)^*, 62\gamma(78\gamma)^*, \gamma(78\gamma)^*]. \end{aligned}$$

The observer-based control $U_{\overline{M}_{opt+}}(\gamma) = \overline{M}_{opt+}\hat{X}(\gamma) \oplus P_{opt+}Q(\gamma)$ with estimated states $\hat{X}(\gamma) = \overline{A}\hat{X}(\gamma) \oplus BU_{\overline{M}_{opt+}}(\gamma) \oplus L_{opt+}(Y(\gamma) \oplus \hat{Y}(\gamma)) = \overline{A}\hat{X}(\gamma) \oplus BU_{\overline{M}_{opt+}}(\gamma) \oplus L_{opt+}Y(\gamma)$ can be realized using a TEG model shown in Fig. 7. The pre-filter P_{opt+}, the observer mapping L_{opt+}, and the state-feedback control M_{opt+} are marked in gray areas. For instance, $L_{opt+}(1,1) = \gamma(78\gamma)^*$ implies that, in the TEG model shown in Fig. 7, there is a cyclic component with one token and 78 time delays for a new transition ξ_3 and the output y has one token and 0 time delay before going through the observer transition $\hat{x}_1$. The other entries of matrix L_{opt+}, the observer-based state-feedback matrix $\overline{M}_{opt+}$ and the prefilter P_{opt+} can be constructed in a similar manner.

The estimated states $\hat{x} = A\hat{x} \oplus Bu_{M_{opt+}} \oplus L_{opt+}y$ in event domain k can be written in the event domain by considering the (max-plus)-algebra as follows:

$$L_{opt+}y : \xi_3(k) = 78\xi_3(k-1) \oplus y(k)$$

$$\hat{x} : \begin{cases} \hat{x}_1(k) = \hat{x}_1(k-1) \oplus \hat{x}_6(k-1) \oplus 1u(k) \oplus \xi_3(k-1), \\ \hat{x}_2(k) = 12\hat{x}_1(k) \oplus 3\hat{x}_3(k) \oplus 12\xi_3(k-1), \\ \hat{x}_3(k) = 9\hat{x}_1(k) \oplus \hat{x}_4(k-1)9\xi_3(k-1), \\ \hat{x}_4(k) = 14\hat{x}_3(k) \oplus 7\hat{x}_5(k) \oplus 23\xi_3(k-1), \\ \hat{x}_5(k) = 7\hat{x}_3(k) \oplus \hat{x}_6(k-1) \oplus 16\xi_3(k-1), \\ \hat{x}_6(k) = 62\hat{x}_5(k) \oplus \xi_3(k). \end{cases}$$

Then the event domain representation for the observer-based control law $u_{Mopt+} = M_{opt+}\hat{x} \oplus P_{opt+}q$ is obtained as follows:

$$\begin{aligned} M_{opt+}\hat{x} :\ & \xi_2(k) = 78\xi_2(k-1) \oplus \hat{x}_1(k) \oplus 66\hat{x}_2(k-1) \\ & \oplus 69\hat{x}_3(k-1) \oplus 55\hat{x}_4(k-1) \oplus 62\hat{x}_5(k-1) \\ & \oplus \hat{x}_6(k-1), \\ P_{opt+}q :\ & \xi_1(k) = 78\xi_1(k-1) \oplus 62q(k-1) \\ v(k)\ & = \xi(k-1) \\ u(k)\ & = \xi_2(k) \oplus v(k), \end{aligned}$$

where ξ_i, $i = 1, 2, 3$, are the intermediate transitions in the TEG shown in Fig. 7. Similarly, the estimated state $\hat{x}$ and the observer-based control law u_{Mopt+} could be written in time-domain equations by considering the min-plus algebra.

The TEG model of the HTS system with open-loop and state feedback controllers are shown in Fig. 7.

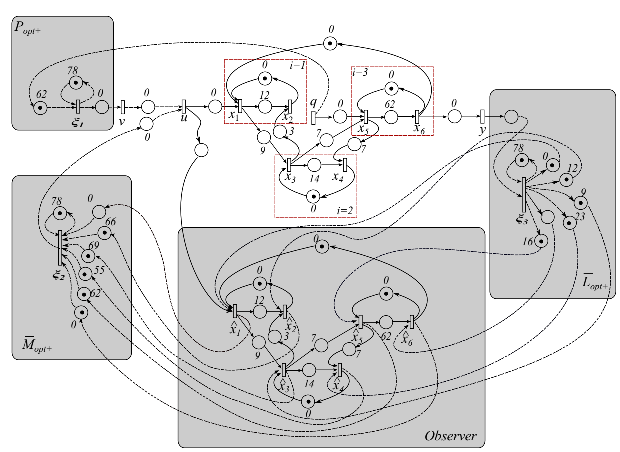

Fig. 7: The TEG model of the HTS system with controllers.

In Fig. 7, the causal pre-filter $P_{opt+} = 62\gamma(78\gamma)^*$ is represented by a cyclic transition due to $(78\gamma)^*$, where γ indicates one token inside of the place and 78 units are the holding time of the token, and 62γ is represented by an additional transition with one token and 62 units holding time. Same analogy works for the feedback $\overline{M}_{opt+} : X \to U$.

When we apply the observer-based control law $U_{\overline{M}_{opt+}}(\gamma) = \overline{M}_{opt+}\hat{X}(\gamma) \oplus P_{opt}Q(\gamma)$ to the system, we can solve the MDDP and the DDP because the observer-based control law preserved the same output as the open-loop controlled system, which solved the DDP because the image inclusion condition $\mathsf{Im}\, C\overline{A}^* S \subset \mathsf{Im}\, C\overline{A}^* B$ since $C\overline{A}^* BP_{opt} = C\overline{A}^* S$. Moreover, based on Proposition 11, the output feedback controller solving the MDDP and DDP generates a smaller controller than the observer based controller because $F_{opt+}C$ is less than $\overline{M}_{opt+}$, where the ouput feedback controller $F_{opt} = -78(78\gamma)^*$ and its causal projection $F_{opt+} = \gamma(78\gamma)^*$, which

$$F_{opt+}C \quad = \quad [\ \varepsilon \quad \varepsilon \quad \varepsilon \quad \varepsilon \quad \varepsilon \quad \gamma(78\gamma)^*\ ] \preceq \overline{M}_{opt+}.$$

VIII. Conclusions

The main contribution of this paper is the design of an observer-based controller solving for the DDP ad MDDP for max-plus linear systems, where only a subset of the states obtained from measurement is available for the controller. This observer-based controller leads to a greater control input than the one obtained with the output feedback strategy based on just-in-time criterion. The main results are illustrated by a better scheduling obtained from the observer-based controller in a high throughput screening system in drug discovery. In conclusion, the estimated state could also be useful in fault detection and diagnosis for max-plus linear systems in future research.

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IX. Appendix

A. Formulas of Star Operations

$a^*(ba^*)^* = (a \oplus b)^* = (a^*b)^*a^*$	(f.1)
$(a^*)^* = a^*$	(f.2)
$(ab)^*a = a(ba)^*$	(f.3)
$a^*a^* = a^*$	(f.4)
$aa^* = a^*a$	(f.5)

B. Formulas of Left Residuations

$a(a \backslash x) \preceq x$	(f.6)
$a \backslash (ax) \succeq x$	(f.7)
$a(a \backslash (ax)) = ax$	(f.8)
$a \backslash (x \wedge y) = a \backslash x \wedge a \backslash y$	(f.9)
$(a \oplus b) \backslash x = a \backslash x \wedge b \backslash x$	(f.10)
$(ab) \backslash x = b \backslash (a \backslash x)$	(f.11)
$b(a \backslash x) \preceq (a / b) \backslash x$	(f.12)
$(a \backslash x)b \preceq a \backslash (xb)$	(f.13)

C. Formulas of Right Residuations

$(x / a)a \preceq x$	(f.14)
$(xa) / a \succeq x$	(f.15)
$((xa) / a)a = xa$	(f.16)
$(x \wedge y) / a = x / a \wedge y / a$	(f.17)
$x / (a \oplus b) = x / a \wedge x / b$	(f.18)
$x / (ba) = (x / a) / b$	(f.19)
$(x / a)b \preceq x \backslash (b / a)$	(f.20)
$b(x / a) \preceq (bx) / a$	(f.21)